O419. Let $x_1, x_2, \ldots, x_n$ be real numbers in the interval $(0, \frac{\pi}{2})$. Prove that

$$\frac{1}{n^2}\left(\frac{\tan x_1}{x_1}+\cdots+\frac{\tan x_n}{x_n}\right)^2 \le \frac{\tan^2 x_1+\cdots+\tan^2 x_n}{{x_1}^2+\cdots+{x_n}^2}.$$

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First note that $\dfrac{\tan x}{x}$ increase in $(0, \pi/2)$.

Indeed

$$\left(\frac{\tan x}{x}\right)' = \frac{x-\sin x\cdot\cos x}{x^2\cos^2 x} = \frac{(x-\sin x)+\sin x\,(1-\cos x)}{x^2\cos^2 x} > 0$$

Then, because n-tiples $\left(x_1^2, ..., x_n^2\right)$ and $\left(\dfrac{\tan^2 x_1}{x_1^2}, ..., \dfrac{\tan^2 x_n}{x_n^2}\right)$ agreed in order, that is $sign\left(x_i^2 - x_j^2\right) = sign\left(\dfrac{\tan^2 x_i}{x_i^2} - \dfrac{\tan^2 x_j}{x_j^2}\right)$ for any $i, j \in \{1, 2, ..., n\}$, by Chebishev's Inequality

$$\sum_{k=1}^{n}\tan^2 x_k = \sum_{k=1}^{n} x_k^2\cdot\frac{\tan^2 x_k}{x_k^2} \ge \sum_{k=1}^{n} x_k^2\cdot\left(\frac{1}{n}\sum_{k=1}^{n}\frac{\tan^2 x_k}{x_k^2}\right).$$

Also, by Quadratic Mean-Arithmetic Mean Inequality

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\tan^2 x_k}{x_k^2} \ge \left(\frac{1}{n}\sum_{k=1}^{n}\frac{\tan x_k}{x_k}\right)^2.$$

Thus,

$$\frac{\sum_{k=1}^{n}\tan^2 x_k}{\sum\limits_{k=1}^{n} x_k^2} \ge \frac{\sum_{k=1}^{n} x_k^2\cdot\left(\frac{1}{n}\sum_{k=1}^{n}\frac{\tan^2 x_k}{x_k^2}\right)}{\sum_{k=1}^{n} x_k^2} \ge \frac{1}{n^2}\left(\sum_{k=1}^{n}\frac{\tan x_k}{x_k}\right)^2.$$

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